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Verification of Optimality and Costate Estimation Using Hilbert Space Projection

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In this paper, we present a direct method for solving optimal control problems based on function approximation using Hilbert space projection. In this approach, the state and control variables are approximated in a finite-dimensional Hilbert space, and the state dynamics and the path constraints are imposed using projection. The resulting nonlinear programming problem is analyzed to derive a set of conditions under which the costates of the optimal control problem can be estimated from the associated Karush–Kuhn–Tucker multipliers. It is shown that the estimated costates have the same order of finite-dimensional approximation as the state and control approximations. The numerical results demonstrate that the present method can address a fairly general class of optimal control problems, including problems with state inequality constraints.

I. Introduction

OPTIMAL control theory has found applications in a variety of fields, including aerospace, engineering, economics, and the sciences. With the availability of increasing computational power, numerical methods to solve optimal control problems have become extremely popular. These methods can largely be divided into two categories: indirect methods and direct methods [1]. In an indirect method, the optimal control problem (OCP) is dualized by introducing the costates, also known as dual variables. The optimality conditions are derived using the calculus of variations and Pontryagin's minimum principle, leading to a two-point boundary-value problem (TPBVP). Various numerical techniques are then used to solve the TPBVP [2]. In a direct method, the optimal control problem is discretized by parameterizing the unknowns to transcribe the continuous-time OCP into a finite-dimensional nonlinear programming problem (NLP) [3]. The NLP is then solved using numerical optimization techniques. Although the indirect methods provide high-accuracy solutions, solving the resulting TPBVP can be extremely difficult due to its sensitivity to the unknown boundary conditions and the initial guess of the costates. Also, the TPBVP has to be derived analytically and, in the presence of path constraints, the switching structure of the constrained/unconstrained arcs has to be known a priori. A direct method requires much less analytical derivation, and the resulting NLP can be solved with relative ease. However, not all direct methods provide information about the optimality of the solution. Verification and validation of results are essential and can be performed naturally by the indirect methods as they are based on solving the necessary conditions derived using the minimum principle. The minimum principle can be used to check the optimality of a solution resulting from a direct method; however, this requires estimates of the dual variables. Therefore, the equivalence between dualization and discretization for a direct method needs to be investigated.

Direct methods for optimal control problems differ in how they approximate the state equations. Most of these methods are based on collocation techniques, in which the constraints are imposed at a set of discrete time instances; popular among these methods are the Hermite–Simpson (HS) and the pseudospectral (PS) methods [4]. The HS methods use piecewise polynomials defined over a few neighboring collocation points and thus are described as local approximation methods [5–7]. The PS methods use globally approximating interpolation polynomials to parameterize the state and control trajectories [8–12] and a form of discretization to represent the state derivatives. In another category of direct methods, known as tau methods [13–17], global orthogonal polynomials are used to parameterize the state and control trajectories and the polynomial coefficients are treated as the optimization variables.

The early motivation to obtain discrete approximations of the costates of an optimal control problem from a direct method stems from the need to generate initial guesses for more accurate indirect methods, like multiple shooting. Martell and Lawton [18] first presented an algorithm to estimate the costates by formulating an auxiliary optimization problem based on the solution obtained from a direct method. Seywald and Kumar [19,20] made use of the sensitivities of the variational cost function to obtain the costate estimates. Other approaches investigated the relationship between the costates and the Karush–Kuhn–Tucker (KKT) multipliers of the NLP. In this framework, Stryk and Bulirsch [21] showed the equivalence between the two for the HS method. Enright and Conway [22] developed an algorithm using a parallel shooting implementation of the Runge–Kutta method to achieve higher accuracy in the costate estimates than that for the HS method. Hager [23] presented the convergence analysis for Runge–Kutta-based direct methods. It was noted that additional conditions on the coefficients of the integration scheme were required for a complete commutation between dualization and discretization. This discrepancy led Hager to design new Runge–Kutta methods for control applications. Ross and Fahroo [11] and Gong et al. [24] presented a similar result for the Legendre pseudospectral method, in which a set of closure conditions are derived under which the costates can be estimated from the KKT mutipliers. Williams [25] generalized the same result for the Jacobi pseudospectral method. More recently, Benson et al. [12] showed equivalence between the discrete costates and the KKT multipliers for the Gauss pseudospectral method. The convergence analysis and costate estimation procedure for tau methods have not been addressed in the literature so far.

In this paper, we present a direct method in a generalized framework of function approximation using Hilbert space projection. In the method of Hilbert space projection (MHSP), the idea is to approximate the state and control variables as linear combinations of a priori selected basis functions of a Hilbert space, whereas the state dynamics and the path constraints are imposed via projection. The MHSP is flexible with respect to the choice of approximating functions, for which both local and global basis functions can be employed. Another main contribution of this paper is the derivation of a costate estimation procedure for the MHSP, which has never

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been reported for the class of tau methods before. We examine the KKT conditions associated with the direct optimization solution and the discretized first-order necessary conditions for the optimal control problem to define a set of "equivalence conditions" under which the two approaches are completely equivalent. In other words, if the equivalence conditions are satisfied, the Euler–Lagrange (indirect) structure of the original OCP is preserved when discretization is performed using the MHSP. In this case, the costate estimates can be obtained from the KKT multipliers of the direct optimization solution. This is a similar result as obtained for the PS methods in [11,26]. In the present context, some interesting observations are made regarding the previously published works on costate estimation for the PS methods.

The paper is organized as follows. In Sec. II, we present a brief introduction to the theory of function approximation and Hilbert space projection. In Sec. III, we define the optimal control problem of interest and some related assumptions. The detailed description of the MHSP is presented in Sec. IV, wherein we derive the nonlinear programming problem and associated KKT first-order necessary conditions. In Sec. V, we derive the equivalence conditions and the costate estimates for the MHSP. In Sec. VI, we describe the MHSP when B-splines are used as approximating functions. Section VII presents remarks on the obtained results with respect to previously published works. In Sec. VIII, we present numerical examples to demonstrate the applicability of the present method.

II. Function Approximation and Hilbert Space Projection

In this section, we first briefly describe the concept of Hilbert space projection applied to find solution of a function approximation problem or a boundary-value problem (BVP). Then we describe our method to transcribe an OCP to an NLP using this technique and discuss the similarities and differences of our approach to the related methods found in the literature. Let $\mathbb{H}$ be a Hilbert space equipped with a norm $\|\cdot\|_{\mathbb{H}}$ and an inner-product $\langle\cdot,\cdot\rangle_{\mathbb{H}}$. Let $\mathbb{V}$ be a closed subspace of $\mathbb{H}$. Then, for any $\mathbf{x}\in\mathbb{H}$, there is an $\hat{\mathbf{x}}\in\mathbb{V}$ such that the error norm $\|\mathbf{e}\|_{\mathbb{H}}=\|\mathbf{x}-\hat{\mathbf{x}}\|_{\mathbb{H}}$ is minimized and for such $\hat{\mathbf{x}}$:

$$\langle\mathbf{e},v\rangle_{\mathbb{H}}=0\quad\forall\ v\in\mathbb{V}\tag{1}$$

This implies that the error $\mathbf{e}$ is orthogonal to the subspace $\mathbb{V}$. Therefore, the projection of $\mathbf{e}$ onto $\mathbb{V}$ is zero.

It is straightforward to solve a function approximation problem using projection. Galerkin and tau methods use projection to solve BVPs [27]. Consider a BVP in the form of Eq. (2):

$$\mathcal{L}(\mathbf{x})=0;\qquad\Delta(\mathbf{x})|_{\text{boundary}}=0\tag{2}$$

where $\mathcal{L}(\cdot)$ is a differential operator, $\mathbf{x}$ is unknown, and $\Delta(\cdot)|_{\text{boundary}}$ is the boundary condition. Let $\mathbb{V}$ be spanned by a linearly independent basis set $B_{\mathbb{V}}:=\{\phi_j\}_{j=1}^N$. In the Galerkin method, the approximate solution $\hat{\mathbf{x}}$ is written as $\hat{\mathbf{x}}=\mathbf{x}_0+\sum_{j=1}^N a_j\phi_j$, where $\mathbf{x}_0$ satisfies the boundary conditions, that is, $\Delta(\mathbf{x}_0)|_{\text{boundary}}=0$, and ϕ_j vanish at the boundary. The coefficients $\{a_j\}_{j=1}^N$ are the unknowns to be solved. Because the exact solution to the problem is not known, the exact error cannot be evaluated. Hence, the projection of error conditions as given in Eq. (1) cannot be used. Therefore, the residual error defined as $\mathcal{R}=\mathcal{L}(\hat{\mathbf{x}})$ is evaluated, and the coefficients a_j are obtained by solving the system of equations generated by setting the projection of $\mathcal{R}$ to zero,

$$\langle\mathcal{R},\phi_j\rangle_{\mathbb{H}}=0\qquad j=1,\ldots,N\tag{3}$$

In the tau method, $B_{\mathbb{V}}$ is chosen to be an orthonormal set and the approximate solution $\hat{\mathbf{x}}$ is not required to satisfy the boundary conditions explicitly as in the case of Galerkin method. The boundary conditions are imposed as equality constraints in terms of the unknown coefficients. In this case, if $\Delta(\cdot)\in\mathbb{R}^p$,

$$\hat{\mathbf{x}}=\sum_{j=1}^N a_j\phi_j$$

the following system of algebraic equations is solved:

$$\Delta(\hat{\mathbf{x}})|_{\text{boundary}}=0\tag{4}$$

$$\langle\mathcal{R},\phi_j\rangle_{\mathcal{H}}=0;\qquad j=1,\ldots,N-p\tag{5}$$

Because p equations are obtained from the boundary condition, a corresponding number of projection equations are not imposed to keep the system of equations deterministic. Thus, to solve for N unknowns $\{a_j\}_{j=1}^N$, p equations are obtained from Eq. (4) and $N-p$ equations are obtained from Eq. (5).

We use a similar idea in the proposed method to solve optimal control problems using Hilbert space projection. In an optimal control problem, the unknown control variables provide extra degrees of freedom to generate a system of algebraic equations. Therefore, the projection equations can be imposed on all basis functions. In our approach, we approximate the state and control variables in a finite-dimensional Hilbert space. The state dynamics and the path constraints are imposed via residual projection on all the basis functions. The boundary conditions are imposed as a set of linear equality constraints in the formulation of the NLP. Because the basis functions are not required to satisfy the boundary conditions explicitly, our method is a close generalization of the tau methods. However, tau methods only use orthogonal approximating functions, such as Legendre [13], Chebyshev [16,17], or Fourier [14,15] series expansions. Our method differs from the tau methods as we require the basis functions only be linearly independent but not necessarily orthogonal. This allows us to use a more general class of basis functions such as B-splines or exponential functions. Depending upon the problem in hand, it can be advantageous to select such basis functions to construct the approximation space.

III. Problem Formulation

For notational simplicity, we consider an optimal control problem in Mayer form. This does not pose any restriction on the present method as any optimal control problem can be represented in Mayer form. We consider a continuous-time autonomous system with mixed state-control constraints and free initial and final times. The objective is to determine the state-control pair $\{\mathbf{x}(\tau)\in\mathbb{R}^n,\mathbf{u}(\tau)\in\mathbb{R}^m;\tau\in[\tau_0,\tau_f]\}$ and time instances τ_0 and τ_f that minimize the cost,

$$J=\Psi(\mathbf{x}(\tau_0),\mathbf{x}(\tau_f),\tau_0,\tau_f)\tag{6}$$

subject to the state dynamics,

$$\dot{\mathbf{x}}(\tau)=F(\mathbf{x}(\tau),\mathbf{u}(\tau))\tag{7}$$

endpoint state equality constraints,

$$\psi(\mathbf{x}(\tau_0),\mathbf{x}(\tau_f),\tau_0,\tau_f)=0\tag{8}$$

and mixed state-control path inequality constraints,

$$\mathbf{h}(\mathbf{x}(\tau),\mathbf{u}(\tau))\le 0\tag{9}$$

where

$$\Psi\colon\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}\to\mathbb{R},\qquad F\colon\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n,$$
$$\psi\colon\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}^p,\qquad \mathbf{h}\colon\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^q$$

are continuously differentiable with respect to their arguments. It is assumed that the optimal solution to this problem exists and, at any time $\tau\in[\tau_0,\tau_f]$, $\frac{\partial\bar{\mathbf{h}}}{\partial\mathbf{u}}$ has full rank, where $\bar{\mathbf{h}}$ is the active constraint set at time τ. Thus, the constraint qualifications required to apply the first-order optimality conditions are implicitly assumed. We consider an autonomous system because an extension to a nonautonomous system is straightforward. This is done by taking time τ as an additional state to the original system and then setting $\dot{\tau}=1$. We consider mixed state-control inequality constraints for simplicity, although numerical results are presented to show that the method of functional approximation is applicable to the problems with pure state inequality constraints also.

IV. Method of Hilbert Space Projection

A direct method to solve an optimal control problem typically consists of two steps. First, state and control trajectories are approximated using a known functional form with a set of coefficients to be determined. Second, a set of equations is derived in terms of the unknown coefficients to impose the state dynamics and an NLP is formulated. In the case of the HS method, for example, the approximation functions are piecewise cubic polynomials with their values at node points as unknown coefficients. The constraints are derived by imposing the state equations at the midpoint of each polynomial segment. In an Nth-order Legendre pseudospectral method, the states are approximated as Nth degree polynomials in Lagrange's form, with their values at the Legendre–Gauss–Lobatto (LGL) points as unknowns. The state equations are also imposed at the LGL points. In this section, we present a direct method to solve problem $\mathcal{B}$ in a generalized framework of function approximation by using Hilbert space projection. In this method, we seek a finite-dimensional approximation of the state and control trajectories in a space of continuous functions $\mathbb{V} := \text{span}\{\phi_1(t), \phi_2(t), \ldots, \phi_N(t), t \in [0, 1]\}$, with a linearly independent basis set $B_{\mathbb{V}} := \{\phi_j(t)\}_{j=1}^N$, and an inner product defined as

$$\langle f, g\rangle = \int_0^1 f(t)g(t)\,\mathrm{d}t; \qquad f(t), g(t) \in \mathbb{V} \tag{10}$$

First, we scale problem $\mathcal{B}$ appropriately so that the state and control trajectories can be approximated in $\mathbb{V}$. Because $\mathbb{V}$ is defined over the time domain [0, 1], the following transformation is used to map the problem from the physical domain $\tau \in [\tau_0, \tau_f]$ to the computational domain $t \in [0, 1]$:

$$\tau(t) = (\tau_f - \tau_0)t + \tau_0 \tag{11}$$

The state and control trajectories are approximated as $\hat{\mathbf{x}}(t), \hat{\mathbf{u}}(t) \in \mathbb{V}$, $t \in [0, 1]$, so that

$$\mathbf{x}\,(\tau(t)) \approx \hat{\mathbf{x}}(t) = \sum_{k=1}^N \alpha_k\phi_k(t) \tag{12}$$

$$\mathbf{u}\,(\tau(t)) \approx \hat{\mathbf{u}}(t) = \sum_{k=1}^N \beta_k\phi_k(t) \tag{13}$$

where $\alpha_k \in \mathbb{R}^n$ and $\beta_k \in \mathbb{R}^m$ are the unknowns. Differentiating the expression in Eq. (12) with respect to τ and using Eq. (11), we get

$$\frac{\mathrm{d}\mathbf{x}(\tau(t))}{\mathrm{d}\tau} \approx \frac{1}{(\tau_f - \tau_0)}\dot{\hat{\mathbf{x}}}(t) = \frac{1}{(\tau_f - \tau_0)}\sum_{k=1}^N \alpha_k\dot{\phi}_k(t) \tag{14}$$

where an overdot denotes the derivative with respect to t. Using Eqs. (7) and (14), we define the residual error in state dynamics as

$$\mathcal{R}_F(t) = (\tau_f - \tau_0)F(\hat{\mathbf{x}}(t), \hat{\mathbf{u}}(t)) - \dot{\hat{\mathbf{x}}}(t) \tag{15}$$

For an exact solution, $\mathcal{R}_F(t)$ is zero everywhere. In collocation-based methods, $\mathcal{R}_F(t)$ is set to zero at a discrete set of points. In the MHSP, the projection of residual error on the approximating Hilbert space $\mathbb{V}$ is set to zero. This is achieved by setting the projection of $\mathcal{R}_F(t)$ on each element of the basis set $B_{\mathbb{V}}$ equal to zero. Using Eq. (10), we write

$$\langle \mathcal{R}_F, \phi_j\rangle = \int_0^1 \mathcal{R}_F(t)\phi_j(t) = 0; \qquad j = 1, 2, \ldots, N \tag{16}$$

Thus, in the present method, the residual error is minimized in an $\mathcal{L}_2$ sense. Next, we introduce a vector of slack variable functions, $\mathbf{s}(t) \in \mathbb{R}^q$, to convert the inequality constraint in Eq. (9) into an equality constraint:

$$\mathbf{h}\,(\mathbf{x}(\tau), \mathbf{u}(\tau)) + \mathbf{s}(\tau) \circ \mathbf{s}(\tau) = 0 \tag{17}$$

where $\circ$ denotes the Hadamard product of two vectors. We approximate the slack variable functions $\mathbf{s}(\tau)$ in the functional space $\mathbb{V}$ so that

$$\mathbf{s}\,(\tau(t)) \approx \hat{\mathbf{s}}(t) = \sum_{k=1}^N \xi_k\phi_k(t) \tag{18}$$

where $\xi_k \in \mathbb{R}^q$. Using Eqs. (12), (13), and (18), the residual in Eq. (17) is obtained as

$$R_h(t) = \mathbf{h}(\hat{\mathbf{x}}(t), \hat{\mathbf{u}}(t)) + \hat{\mathbf{s}}(t) \circ \hat{\mathbf{s}}(t) \tag{19}$$

Next, we set the projection of $R_h(t)$ on each element of the basis set $B_{\mathbb{V}}$ equal to zero:

$$(\tau_f - \tau_0)\int_0^1 R_h(t)\phi_j(t) = 0; \qquad j = 1, 2, \ldots, N \tag{20}$$

As will become clear in later sections, Eq. (20) is scaled by $(\tau_f - \tau_0)$ to obtain the costate estimates in a simpler form. This scaling does not alter the value of the constraint in Eq. (17). Finally, the endpoint equality constraint Eq. (8) is imposed as

$$\psi(\hat{\mathbf{x}}(0), \hat{\mathbf{x}}(1), \tau_0, \tau_f) = 0 \tag{21}$$

Next, the discretization method presented herein is used to transcribe the optimal control problem into a nonlinear programming problem.

A. Nonlinear Programming Problem: B_ϕ

Function approximation of state and control trajectories using Eqs. (12) and (13), combined with the Hilbert space projection as in Eqs. (16), (20), and (21), transcribe problem $\mathcal{B}$ into a finite-dimensional nonlinear programming problem, denoted as problem B_ϕ. For the subsequent treatment, we denote the approximate state dynamics as

$$\hat{F}(\alpha_k, \beta_k, \phi_k(t)) = F(\hat{\mathbf{x}}(t), \hat{\mathbf{u}}(t)) \tag{22}$$

Using the same notation for all other functionals, problem B_ϕ is to determine $\{\alpha_k \in \mathbb{R}^n, \beta_k \in \mathbb{R}^m, \xi_k \in \mathbb{R}^q\}_{k=1}^N$ and time instances τ_0 and τ_f that minimize the cost,

$$\hat{J} = \hat{\Psi}(\alpha_k, \phi_k(0), \phi_k(1), \tau_0, \tau_f) \tag{23}$$

subject to the constraints,

$$\int_0^1\left[(\tau_f - \tau_0)\hat{F}(\alpha_k, \beta_k, \phi_k(t)) - \sum_{k=1}^N \alpha_k\dot{\phi}_k(t)\right]\phi_j(t)\,\mathrm{d}t = 0; \quad j = 1, \ldots, N \tag{24}$$

$$\hat{\psi}(\alpha_k, \phi_k(0), \phi_k(1), \tau_0, \tau_f) = 0 \tag{25}$$

$$(\tau_f - \tau_0)\int_0^1\left[\hat{\mathbf{h}}(\alpha_k, \beta_k, \phi_k(t)) + \sum_{k=1}^N \xi_k\phi_k(t) \circ \sum_{k=1}^N \xi_k\phi_k(t)\right]\phi_j(t)\,\mathrm{d}t = 0; \quad j = 1, \ldots, N \tag{26}$$

Problem B_ϕ constituting Eqs. (23–26) can be solved using standard numerical optimization software. Any numerical quadrature scheme can be used to evaluate the integral expressions in Eqs. (23), (24), and (26). In some cases, depending upon the problem in hand, the integrals can be evaluated exactly in symbolic form by using available software tools. For the results presented in this paper, we use SNOPT [28] as the optimization solver and MATLAB®'s Symbolic Math Toolbox for integral evaluations. Next, to facilitate the derivation of the costate estimation procedure and related equivalence conditions, we derive the KKT first-order necessary conditions associated with problem B_ϕ.

B. Karush–Kuhn–Tucker Conditions for Problem B_ϕ: $B_{\phi\lambda}$

The Lagrangian for problem B_ϕ is formed by adjoining the cost function with the constraint equations, written as

$$\begin{aligned} J' &= \hat{\Psi}(\alpha_k, \phi_k(0), \phi_k(1), \tau_0, \tau_f) \\ &+ \sum_{j=1}^{N} \gamma_j^T \int_0^1 \bigg[(\tau_f - \tau_0)\hat{F}(\alpha_k, \beta_k, \phi_k(t)) \\ &- \sum_{k=1}^{N} \alpha_k \dot{\phi}_k(t) \bigg] \phi_j(t)\,\mathrm{d}t \\ &+ (\tau_f - \tau_0) \sum_{j=1}^{N} \eta_j^T \int_0^1 \bigg[\hat{\mathbf{h}}(\alpha_k, \beta_k, \phi_k(t)) \\ &+ \sum_{k=1}^{N} \xi_k \phi_k(t) \circ \sum_{l=1}^{N} \xi_l \phi_l(t) \bigg] \phi_j(t)\,\mathrm{d}t \\ &+ \nu^T \hat{\psi}(\alpha_k, \phi_k(0), \phi_k(1), \tau_0, \tau_f) \end{aligned} \tag{27}$$

where $\gamma_j \in \mathbb{R}^n$, $\eta_j \in \mathbb{R}^q$, and $\nu \in \mathbb{R}^p$ are the KKT multipliers associated with the constraints given by Eqs. (24), (26), and (25), respectively. The KKT first-order necessary conditions are obtained by setting the derivatives of the Lagrangian J' with respect to the unknowns $\{\alpha_i, \beta_i, \gamma_i, \eta_i, \xi_i, \nu, t_0, t_f\}$ equal to zero. For brevity, we introduce the following shorthand notation:

1) We use $\hat{F}$ to denote $\hat{F}(\alpha_k, \beta_k, \phi_k(t))$.

2) We use a subscript to denote a partial derivative, for example, $\frac{\partial}{\partial \mathbf{x}} F(\mathbf{x}, \mathbf{u})$ is written as $F_\mathbf{x}$. Using the same notation for all other variables, we have for $i = 1, \ldots, N$,

$$\begin{aligned} \frac{\partial J'}{\partial \alpha_i} &= (\tau_f - \tau_0) \int_0^1 \bigg[\hat{F}_\mathbf{x}^T \sum_{j=1}^{N} \gamma_j \phi_j(t) + \hat{\mathbf{h}}_\mathbf{x}^T \sum_{j=1}^{N} \eta_j \phi_j(t) \bigg] \phi_i(t)\,\mathrm{d}t \\ &- \sum_{j=1}^{N} \int_0^1 \gamma_j \dot{\phi}_i(t) \phi_j(t)\,\mathrm{d}t + [\hat{\Psi}_{\mathbf{x}(0)} + \hat{\psi}_{\mathbf{x}(0)}^T \nu] \phi_i(0) + [\hat{\Psi}_{\mathbf{x}(1)} \\ &+ \hat{\psi}_{\mathbf{x}(1)}^T \nu] \phi_i(1) \end{aligned} \tag{28}$$

Using integration by parts, we write

$$\begin{aligned} \sum_{j=1}^{N} \int_0^1 \gamma_j \dot{\phi}_i(t) \phi_j(t)\,\mathrm{d}t &= \sum_{j=1}^{N} [\phi_i(1)\phi_j(1) - \phi_i(0)\phi_j(0)] \\ &- \sum_{j=1}^{N} \int_0^1 \gamma_j \dot{\phi}_j(t) \phi_i(t)\,\mathrm{d}t \end{aligned} \tag{29}$$

Using Eqs. (28) and (29) and rearranging, we get

$$\begin{aligned} 0 &= (\tau_f - \tau_0) \int_0^1 \bigg[\hat{F}_\mathbf{x}^T \sum_{j=1}^{N} \gamma_j \phi_j(t) + \hat{\mathbf{h}}_\mathbf{x}^T \sum_{j=1}^{N} \eta_j \phi_j(t) \\ &+ \frac{1}{(\tau_f - \tau_0)} \sum_{j=1}^{N} \gamma_j \dot{\phi}_j(t) \bigg] \phi_i(t)\,\mathrm{d}t \\ &+ \bigg[\hat{\Psi}_{\mathbf{x}(0)} + \hat{\psi}_{\mathbf{x}(0)}^T \nu + \sum_{j=1}^{N} \gamma_j \phi_j(0) \bigg] \phi_i(0) \\ &+ \bigg[\hat{\Psi}_{\mathbf{x}(1)} + \hat{\psi}_{\mathbf{x}(1)}^T \nu - \sum_{j=1}^{N} \gamma_j \phi_j(1) \bigg] \phi_i(1) \end{aligned} \tag{30}$$

also,

$$\frac{\partial J'}{\partial \beta_i} = (\tau_f - \tau_0) \int_0^1 \bigg[\hat{F}_\mathbf{u}^T \sum_{j=1}^{N} \gamma_j \phi_j(t) + \hat{\mathbf{h}}_\mathbf{u}^T \sum_{j=1}^{N} \eta_j \phi_j(t) \bigg] \phi_i(t)\,\mathrm{d}t = 0 \tag{31}$$

$$\frac{\partial J'}{\partial \gamma_i} = \int_0^1 \bigg[(\tau_f - \tau_0)\hat{F} - \sum_{k=1}^{N} \alpha_k \dot{\phi}_k(t) \bigg] \phi_i(t)\,\mathrm{d}t = 0 \tag{32}$$

$$\frac{\partial J'}{\partial \eta_i} = \int_0^1 \bigg[\hat{\mathbf{h}} + \sum_{k=1}^{N} \xi_k \phi_k(t) \circ \sum_{l=1}^{N} \xi_l \phi_l(t) \bigg] \phi_i(t)\,\mathrm{d}t = 0 \tag{33}$$

$$\frac{\partial J'}{\partial \xi_i} = 2 \int_0^1 \bigg[\sum_{k=1}^{N} \xi_k \phi_k(t) \circ \sum_{j=1}^{N} \eta_j \phi_j(t) \bigg] \phi_i(t)\,\mathrm{d}t = 0 \tag{34}$$

$$\frac{\partial J'}{\partial \nu} = \hat{\psi} = 0 \tag{35}$$

$$\frac{\partial J'}{\partial \tau_0} = \int_0^1 \bigg[\hat{F}^T \sum_{j=1}^{N} \gamma_j \phi_j(t) \bigg] \mathrm{d}t - [\hat{\Psi}_{\tau_0} + \nu^T \hat{\psi}_{\tau_0}] = 0 \tag{36}$$

$$\frac{\partial J'}{\partial \tau_f} = \int_0^1 \bigg[\hat{F}^T \sum_{j=1}^{N} \gamma_j \phi_j(t) \bigg] \mathrm{d}t + [\hat{\Psi}_{\tau_f} + \nu^T \hat{\psi}_{\tau_f}] = 0 \tag{37}$$

Equations (30–37) constitute the KKT conditions for problem $\mathcal{B}_\phi$. Next, we derive the first-order optimality conditions for problem $\mathcal{B}$, which are then discretized to derive equivalence between the costates of the optimal control problem and the KKT multipliers of the associated NLP.

V. Costate Estimation

As stated earlier, Fahroo and Ross first derived a set of conditions under which a linear mapping exists between the costates and the KKT multipliers of the nonlinear programming problem for the Legendre pseudospectral method. In a similar fashion, Williams [25] has derived a costate estimation procedure for nonclassical pseudospectral methods. More rigorously, their derivation is based on the commutative nature of problems $B^{\lambda N}$ and $B^{N\lambda}$ under a set of closure conditions, for which problem $B^{N\lambda}$ is the set of KKT conditions associated with the NLP and problem $B^{\lambda N}$ is the set of discretized first-order optimality conditions. We adopt a similar approach by comparing problems $\mathcal{B}_{\lambda\phi}$ and $\mathcal{B}_{\phi\lambda}$ to derive equivalence conditions under which these two problems commute.

A. First-Order Optimality Conditions for Problem $\mathcal{B}$: $\mathcal{B}_\lambda$

Problem $\mathcal{B}$ can be solved by applying a calculus of variations and Pontryagin's minimum principle. In this framework, the first-order necessary conditions for optimality lead to a two-point boundary-value problem derived by using the augmented Hamiltonian $\mathcal{H}$ and the terminal cost $\mathcal{C}$ defined as

$$\mathcal{H}(\mathbf{x}, \mathbf{u}, \lambda, \mu, \mathbf{s}) = \lambda^T(\tau) F(\mathbf{x}, \mathbf{u}) + \mu^T(\tau)[\mathbf{h}(\mathbf{x}, \mathbf{u}) + \mathbf{s}(\tau) \circ \mathbf{s}(\tau)] \tag{38}$$

$$\begin{aligned} \mathcal{C}(\mathbf{x}(\tau_0), \mathbf{x}(\tau_f), \nu, \tau_0, \tau_f) &= \Psi(\mathbf{x}(\tau_0), \mathbf{x}(\tau_f), \tau_0, \tau_f) \\ &+ \upsilon^T \psi(\mathbf{x}(\tau_0), \mathbf{x}(\tau_f), \tau_0, \tau_f) \end{aligned} \tag{39}$$

where $\lambda(\tau) \in \mathbb{R}^n$ is the costate; $\mu(\tau) \in \mathbb{R}^q$ and $\upsilon \in \mathbb{R}^p$ are the Lagrange multipliers; and $\mathbf{s}(\tau) \in \mathbb{R}^q$ is the slack variable function. The time dependence of state and control trajectories has been dropped for brevity. Problem $\mathcal{B}_\lambda$ seeks to find the functions $\{\mathbf{x}(\tau), \mathbf{u}(\tau), \lambda(\tau), \mu(\tau); \tau \in [\tau_0, \tau_f]\}$, vector υ, and time instances τ_0 and τ_f that satisfy the following conditions:

$$\begin{aligned} &\dot{\mathbf{x}}(\tau) = F(\mathbf{x}, \mathbf{u}) \\ &\dot{\lambda}(\tau) + \mathcal{H}_\mathbf{x} = \dot{\lambda}(\tau) + F_\mathbf{x}^T \lambda(\tau) + \mathbf{h}_\mathbf{x}^T \mu(\tau) = 0 \\ &\mathcal{H}_\mathbf{u} = F_\mathbf{u}^T \lambda(\tau) + \mathbf{h}_\mathbf{u}^T \mu(\tau) = 0, \qquad \mathbf{h}(\mathbf{x}, \mathbf{u}) + \mathbf{s}(\tau) \circ \mathbf{s}(\tau) = 0 \\ &\mathcal{H}_\mathbf{s} = 2\mu(\tau) \circ \mathbf{s}(\tau) = 0, \qquad \psi(\mathbf{x}(\tau_0), \mathbf{x}(\tau_f), \tau_0, \tau_f) = 0 \\ &\{\lambda(\tau_0), \lambda(\tau_f)\} = \{-\mathcal{C}_{\mathbf{x}(\tau_0)}, \mathcal{C}_{\mathbf{x}(\tau_f)}\} \\ &\{\mathcal{H}|_{\tau=\tau_0}, \mathcal{H}|_{\tau=\tau_f}\} = \{\mathcal{C}_{\tau_0}, -\mathcal{C}_{\tau_f}\} \end{aligned} \tag{40}$$

B. Discretized First-Order Optimality Conditions: $\mathcal{B}_{\lambda\phi}$

Problem $\mathcal{B}_\lambda$ as defined by equation set (40) must be discretized to obtain conditions for optimality in the functional space $\mathbb{V}$. The costate trajectories and the Lagrange multipliers are approximated as

$$\lambda(\tau(t)) \approx \hat{\lambda}(t) = \sum_{k=1}^{N} \tilde{\gamma}_k \phi_k(t) \tag{41}$$

$$\dot{\lambda}(\tau(t)) \approx \frac{1}{(\tau_f - \tau_0)} \dot{\hat{\lambda}}(t) = \frac{1}{(\tau_f - \tau_0)} \sum_{k=1}^{N} \tilde{\gamma}_k \dot{\phi}_k(t) \tag{42}$$

$$\mu(\tau(t)) \approx \hat{\mu}(t) = \sum_{k=1}^{N} \tilde{\eta}_k \phi_k(t) \tag{43}$$

where $\tilde{\gamma}_k \in \mathbb{R}^n$ and $\tilde{\eta}_k \in \mathbb{R}^q$. Using Eqs. (12), (13), (18), and (41–43), we obtain the residual error $\mathcal{R}_\lambda$ for equation set (40). We set the projection of $\mathcal{R}_\lambda$ on each element of the basis set $B_{\mathbb{V}}$ equal to zero. The following algebraic equations are obtained for $i = 1, \ldots, N$:

$$\int_0^1 \left[(\tau_f - \tau_0)\hat{F} - \sum_{k=1}^{N} \alpha_k \dot{\phi}_k(t) \right] \phi_i(t)\, \mathrm{d}t = 0 \tag{44}$$

$$\int_0^1 \left[\hat{F}_{\mathbf{x}}^T \sum_{k=1}^{N} \tilde{\gamma}_k \phi_k(t) + \hat{\mathbf{h}}_{\mathbf{x}}^T \sum_{k=1}^{N} \tilde{\eta}_k \phi_k(t) + \frac{1}{(\tau_f - \tau_0)} \sum_{k=1}^{N} \tilde{\gamma}_k \dot{\phi}_k(t) \right] \phi_i(t)\, t = 0 \tag{45}$$

$$\int_0^1 \left[\hat{F}_{\mathbf{u}}^T \sum_{k=1}^{N} \tilde{\gamma}_k \phi_k(t) + \hat{\mathbf{h}}_{\mathbf{u}}^T \sum_{k=1}^{N} \tilde{\eta}_k \phi_k(t) \right] \phi_i(t)\, \mathrm{d}t = 0 \tag{46}$$

$$\int_0^1 \left[\hat{\mathbf{h}} + \sum_{k=1}^{N} \xi_k \phi_k(t) \circ \sum_{l=1}^{N} \xi_l \phi_l(t) \right] \phi_i(t)\, \mathrm{d}t = 0 \tag{47}$$

$$2\int_0^1 \left[\sum_{k=1}^{N} \xi_k \phi_k(t) \circ \sum_{k=1}^{N} \tilde{\eta}_k \phi_k(t) \right] \phi_i(t)\, \mathrm{d}t = 0 \tag{48}$$

$$\hat{\psi}(\hat{\mathbf{x}}(\tau_0), \hat{\mathbf{x}}(\tau_f), \tau_0, \tau_f) = 0 \tag{49}$$

$$\left\{ \sum_{k=1}^{N} \tilde{\gamma}_k \phi_k(0), \sum_{k=1}^{N} \tilde{\gamma}_k \phi_k(1) \right\} = \{-\mathcal{C}_{\mathbf{x}(0)}, \mathcal{C}_{\mathbf{x}(1)}\} \tag{50}$$

$$\{\hat{\mathcal{H}}|_{t=0}, \hat{\mathcal{H}}|_{t=1}\} = \{\mathcal{C}_{\tau_0}, -\mathcal{C}_{\tau_f}\} \tag{51}$$

where $i = 1, 2, \ldots, N$. Equations (44–51) represent the indirect method solution to problem $\mathcal{B}$, and their solution is an approximation of the true optimal solution in the functional space $\mathbb{V}$. Next, we compare the discretized optimality conditions $\mathcal{B}_{\lambda\phi}$ with the KKT conditions $\mathcal{B}_{\phi\lambda}$ to obtain the equivalence between the costate functions and the KKT multipliers of problem $\mathcal{B}_\phi$.

C. Equivalence Conditions

We define a set of conditions that are necessary for the equivalence of the results obtained by solving problems $\mathcal{B}_{\lambda\phi}$ and $\mathcal{B}_{\phi\lambda}$. Comparing Eqs. (30) and (45), a complete equivalence of the two conditions requires

$$\sum_{j=1}^{N} \gamma_j \phi_j(0) = -[\hat{\Psi}_{\mathbf{x}(0)} + \hat{\psi}_{\mathbf{x}(0)}^T \nu] \tag{52}$$

$$\sum_{j=1}^{N} \gamma_j \phi_j(1) = [\hat{\Psi}_{\mathbf{x}(1)} + \hat{\psi}_{\mathbf{x}(1)}^T \nu] \tag{53}$$

Comparing Eqs. (36) and (37) with Eq. (51), we obtain

$$\hat{\mathcal{H}}|_{t=0} = \hat{\mathcal{H}}|_{t=1} = \int_0^1 \left[\hat{F} \sum_{j=1}^{N} \gamma_j^T \phi_j(t) \right] \mathrm{d}t \tag{54}$$

Thus, Eqs. (52–54), when added to the KKT conditions, fill the "gap" between the direct and indirect method solutions of problem $\mathcal{B}$ (see Fig. 1). We see that the direct method discretization of problem $\mathcal{B}$ does not explicitly impose the boundary conditions on the discrete costates. This loss of information is restored by the equivalence conditions in Eqs. (52) and (53). The condition given by Eq. (54) states the constancy of the Hamiltonian for an autonomous system.

D. Costate Estimates

The equivalence conditions defined in the previous subsection provide the mapping between KKT multipliers $\{\gamma_k\}$ and the costate approximation $\hat{\lambda}(t)$. Similarly, a mapping exists between the KKT multipliers $\{\eta_k\}$ and the approximation of the Lagrange multiplier function $\hat{\eta}(t)$. Thus, under the conditions that Eqs. (52–54) hold, comparing $\mathcal{B}_{\lambda\phi}$ and $\mathcal{B}_{\phi\lambda}$, we get

$$\gamma_k = \tilde{\gamma}_k, \qquad \eta_k = \tilde{\eta}_k; \qquad k = 1, \ldots, N \tag{55}$$

$$\upsilon = \nu \tag{56}$$

Using Eqs. (55) and (56), the costates can be estimated in the Hilbert space $\mathbb{V}$. We summarize the costate estimation results for the MHSP via the following theorem.

Theorem 1: Costate Mapping Theorem for the MHSP: Assume that an optimal control problem is solved using the MHSP with an underlying finite-dimensional Hilbert space $\mathbb{V}$ and the equivalence conditions hold. Then, the $\mathcal{L}_2$-optimal estimates of the costates $(\hat{\lambda}(t), \hat{\mu}(t))$ in $\mathbb{V}$ and the terminal covector (υ) can be obtained using the KKT multipliers (γ_k, η_k, ν) of the associated NLP as

$$\lambda(\tau(t)) \approx \hat{\lambda}(t) = \sum_{k=1}^{N} \gamma_k \phi_k(t) \tag{57}$$

$$\mu(\tau(t)) \approx \hat{\mu}(t) = \sum_{k=1}^{N} \eta_k \phi_k(t) \tag{58}$$

$$\upsilon = \nu \tag{59}$$

Proof: The solution to problem $\mathcal{B}_{\lambda\phi}$ exists by assumption. Because equivalence conditions hold, substitution of Eq. (55) into Eqs. (41) and (43) completes the proof.

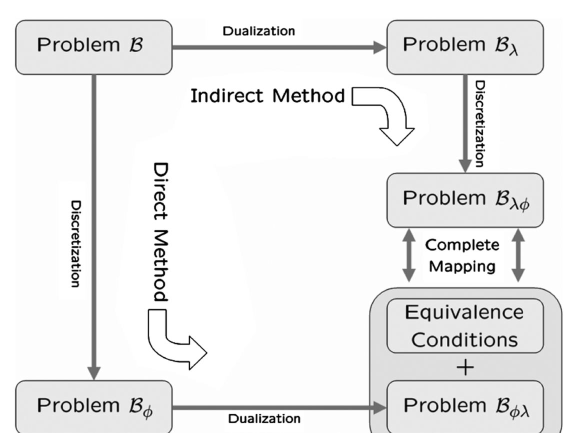


Fig. 1 For a complete mapping to exist between $\mathcal{B}_\phi$ (direct method) and $\mathcal{B}_\lambda$ (indirect method), a set of equivalence conditions must be satisfied along with the KKT conditions ($\mathcal{B}_{\phi\lambda}$) associated with $\mathcal{B}_\phi$.

VI. Method of Hilbert Space Projection Using B-Spline Approximation

There are a number of ways to select the basis functions for state and control parameterization. The basis functions should be linearly independent and continuous on the computational domain of the problem in hand. In this section, we describe an algorithm to solve optimal control problems using B-spline approximations and the MHSP. A B-spline is a piecewise polynomial function with a specified level of global smoothness. Also, the B-spline basis functions have local support, which means that each basis function only influences a local region of the global trajectory. Local support is a desirable property of basis functions for numerically stable algorithms. We first give a brief introduction of the construction and properties of B-splines. Then we describe the implementation of the MHSP on a computer.

A. B-Spline Approximation

A B-spline is a function defined on the interval $[t_0, t_f]$ of the real line, composed of segments of polynomials that are stitched at predefined break points satisfying a given degree of smoothness. The break points are a strictly increasing set of points $\{\zeta_i\}_{i=0}^N$ such that $t_0 = \zeta_0 < \zeta_1 < \cdots < \zeta_N = t_f$. The number of continuous derivatives across the breakpoints defines the order of smoothness. An order of smoothness s_i at a breakpoint ζ_i implies that the curve is C^{s_i-1} continuously differentiable at ζ_i. Given the number of subintervals (N), the order of each polynomial segment (r), and the order of smoothness (s) at the breakpoints, a B-spline curve $y(t)$ is represented in the basis form as

$$y(t) = \sum_{k=1}^{N_c} \alpha_k B_{k,r}(t) \tag{60}$$

where α_k are the free parameters, and $N_c = N(r-s) + s$ is the number of free parameters or the degrees of freedom of $y(t)$. To construct the basis functions $B_{k,r}(t)$, we define a knot vector. A knot vector Γ is a nondecreasing sequence containing breakpoints with a multiplicity of $(r-s)$ at the interior breakpoints. The multiplicity of end points $\{\zeta_0, \zeta_N\}$ is r.

$$\begin{aligned}\Gamma &= [c_1, c_2, \ldots, c_{(N_c+r)}] \\ &= [\overbrace{\zeta_0, \zeta_0, \ldots, \zeta_0}^{r-\text{times}} \ldots \overbrace{\zeta_i, \zeta_i, \ldots, \zeta_i}^{(r-s)-\text{times}} \ldots \overbrace{\zeta_N, \zeta_N, \ldots, \zeta_N}^{r-\text{times}}]\end{aligned} \tag{61}$$

The basis functions $B_{k,r}(t)$ are defined by a recurrence relationship [29]:

$$B_{k,0}(t) = \begin{cases} 1, & \text{if } c_i \le t < c_{i+1}; \\ 0, & \text{otherwise} \end{cases}$$

$$B_{k,r}(t) = \frac{t - c_k}{c_{k+r+1} - c_k} B_{k,r-1}(t) + \frac{c_{k+r} - t}{c_{k+r} - c_{k+1}} B_{k+1,r-1}(t) \tag{62}$$

The B-spline basis functions defined by Eq. (62) are continuous to a specified degree, have local support, and are linearly independent. A comprehensive list of B-spline properties can be found in [29]. Figure 2 provides an example of a B-spline and its basis functions with $N = 4$, $r = 4$, and $s = 3$.

B. Numerical Implementation of the Method of Hilbert Space Projection

In this subsection, we describe the numerical implementation of the MHSP, which requires that the integrals in Eqs. (24) and (26) be evaluated numerically. This can be accomplished by using a numerical quadrature scheme. Using numerical quadrature changes the structure of the resulting NLP and, in turn, the KKT conditions. However, the results on costate estimation and equivalence conditions can still be derived if the quadrature scheme is chosen so that the integration by parts formula in Eq. (29) holds. Here we take a simple example to substantiate this claim in the case of B-spline approximation. Consider an optimal control problem $\mathcal{A}$ to determine the state-control pair $\{\mathbf{x}(t) \in \mathbb{R}^n, \mathbf{u}(t) \in \mathbb{R}^m; t \in [0, 1]\}$ that minimizes

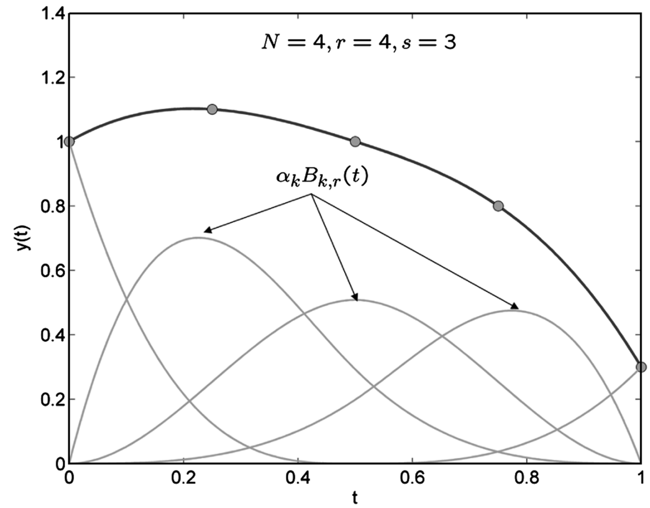


Fig. 2 Spline curve as a combination of B-splines.

$$J = \Psi(\mathbf{x}(1)) \tag{63}$$

subject to

$$\dot{\mathbf{x}}(t) = F(\mathbf{x}(t), \mathbf{u}(t)) \tag{64}$$

$$\psi(\mathbf{x}(0), \mathbf{x}(1)) = 0 \tag{65}$$

where Ψ: $\mathbb{R}^n \to \mathbb{R}$, F: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and ψ: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$. To solve this problem using the MHSP, we approximate the state and control trajectories as B-splines. Thus,

$$\mathbf{x}\,(t) \approx \hat{\mathbf{x}}(t) = \sum_{k=1}^{N} \alpha_k B_{k,r}(t), \quad \mathbf{u}(t) \approx \hat{\mathbf{u}}(t) = \sum_{k=1}^{N} \beta_k B_{k,r}(t) \tag{66}$$

for some N, r, s, and breakpoints $0 = t_0 < t_1 < \ldots < t_N = 1$. Let $\Omega_i := [t_{i-1}, t_i]$. For each domain Ω_i, let $\{b_{ij}\}_{j=0}^{N_q}$ be the number N_q of LGL quadrature points with corresponding quadrature weights w_{ij} (see Fig. 3). Here we assume that the weights w_{ij} have been appropriately scaled to transform the integration domain $[-1, 1]$ of the LGL points to Ω_i. Then, the integral of a function $f(t)$ over interval Ω_i can be approximated as

$$\int_{\Omega_i} f(t)\,\mathrm{d}t \approx \sum_{j=0}^{N_q} w_{ij} f(b_{ij}) \tag{67}$$

Also, given N_q, the integral evaluation using the LGL rule is exact for polynomials of degree $2N_q - 1$ or less:

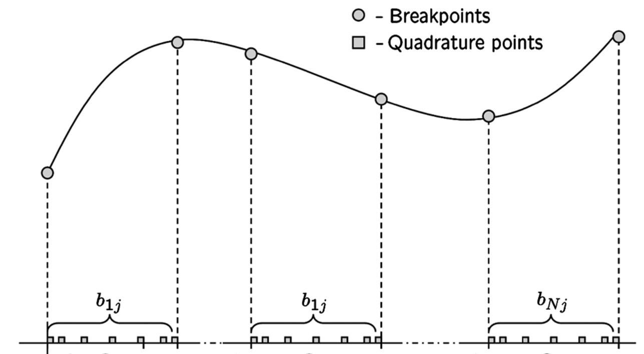


Fig. 3 Arrangement of breakpoints and quadrature points in domain $t \in [0, 1]$. $\{t_i\}_{i=0}^N$ are the breakpoints. $\{b_{ij}\}_{j=0}^{N_q}$ are the quadrature points for domain $\Omega_i = [t_{i-1}, t_i]$.

$$p(t) \in P^{2N_q-1}; \qquad \int_{\Omega_i} p(t)\,\mathrm{d}t = \sum_{j=0}^{N_q} w_{ij} p(b_{ij}) \tag{68}$$

where P^n represents the space of all polynomials of a degree less than or equal to n. The B-spline basis functions are piecewise polynomials such that $\{B_{k,r}(t) \in P^{r-1}; t \in \Omega_i\}_{k=1}^N$ over domains $\{\Omega_i\}_{i=1}^N$. We choose $N_q \geq r-1$, for which the following integration by parts formula holds:

$$\sum_{i=1}^{N}\sum_{j=0}^{N_q} w_{ij}[\dot{B}_{k,r}(b_{ij})B_{l,r}(b_{ij}) + \dot{B}_{l,r}(b_{ij})B_{k,r}(b_{ij})] = [B_{k,r}(1)B_{l,r}(1) - B_{k,r}(0)B_{l,r}(0)] \tag{69}$$

A nonlinear programming problem, $\mathcal{A}_N$, is formulated based on Sec. IV.A and using Eq. (67) to approximate the integrals. The problem $\mathcal{A}_N$ is to determine $\{\alpha_k \in \mathbb{R}^n, \beta_k \in \mathbb{R}^m\}_{k=1}^N$ that minimize

$$J = \hat{\Psi}(\alpha_k, B_{k,r}(1)) \tag{70}$$

subject to the constraints

$$\sum_{i=1}^{N}\left\{\sum_{j=0}^{N_q} w_{ij}\left[\hat{F}(\alpha_k, \beta_k, B_{k,r}(b_{ij})) - \sum_{k=1}^{N}\alpha_k \dot{B}_{k,r}(b_{ij})\right]B_{l,r}(b_{ij})\right\} = 0 \qquad l = 1, 2, \ldots, N \tag{71}$$

$$\hat{\psi}(\alpha_k, B_{k,r}(0), B_{k,r}(1)) = 0 \tag{72}$$

Problem $\mathcal{A}_N$ can be solved using standard numerical optimization software. Next, we derive the KKT conditions for problem $\mathcal{A}_N$. The augmented cost is defined as

$$J' = \hat{\Psi}(\alpha_k, B_{k,r}(1)) + \sum_{l=1}^{N}\gamma_l\left[\sum_{i=1}^{N}\left\{\sum_{j=0}^{N_q} w_{ij}\left[\hat{F}(\alpha_k, \beta_k, B_{k,r}(b_{ij})) - \sum_{k=1}^{N}\alpha_k\dot{B}_{k,r}(b_{ij})\right]B_{l,r}(b_{ij})\right\}\right] + \nu^T\hat{\psi}(\alpha_k, B_{k,r}(0), B_{k,r}(1)) \tag{73}$$

where $\gamma_l \in \mathbb{R}^n$ and $\nu \in \mathbb{R}^p$ are the KKT multipliers associated with the constraints given by Eqs. (71) and (72), respectively. The KKT conditions denoted by $\mathcal{A}_{N\lambda}$ are derived by setting the partial derivatives of J' with respect to free variables equal to zero. Thus, for $m = 1, \ldots, N$,

$$0 = \frac{\partial J'}{\partial \alpha_m} = \sum_{l=1}^{N}\gamma_l\left[\sum_{i=1}^{N}\left\{\sum_{j=0}^{N_q} w_{ij}[\hat{F}_{\mathbf{x}}B_{m,r}(b_{ij})B_{l,r}(b_{ij}) - \dot{B}_{m,r}(b_{ij})B_{l,r}(b_{ij})]\right\}\right] \tag{74}$$

$$+ [\hat{\Psi}_{\mathbf{x}(1)} + \hat{\psi}_{\mathbf{x}(1)}\nu]B_{m,r}(1) + \hat{\psi}_{\mathbf{x}(0)}\nu B_{m,r}(0) \tag{75}$$

Using Eq. (69), we get

$$0 = \frac{\partial J'}{\partial \alpha_m} = \sum_{l=1}^{N}\gamma_l\left[\sum_{i=1}^{N}\left\{\sum_{j=0}^{N_q} w_{ij}[\hat{F}_{\mathbf{x}}B_{m,r}(b_{ij})B_{l,r}(b_{ij}) + \dot{B}_{l,r}(b_{ij})B_{m,r}(b_{ij})]\right\}\right] - \sum_{l=1}^{N}\gamma_l[B_{m,r}(1)B_{l,r}(1) - B_{m,r}(0)B_{l,r}(0)] + [\hat{\Psi}_{\mathbf{x}(1)} + \hat{\psi}_{\mathbf{x}(1)}\nu]B_{m,r}(1) + \hat{\psi}_{\mathbf{x}(0)}\nu B_{m,r}(0) \tag{76}$$

Rearranging Eq. (76),

$$0 = \frac{\partial J'}{\partial \alpha_m} = \sum_{i=1}^{N}\left\{\sum_{j=0}^{N_q} w_{ij}\left[\hat{F}_{\mathbf{x}}\sum_{l=1}^{N}\gamma_l B_{l,r}(b_{ij}) + \sum_{l=1}^{N}\gamma_l\dot{B}_{l,r}(b_{ij})\right]B_{m,r}(b_{ij})\right\} + \left[\hat{\Psi}_{\mathbf{x}(1)} + \hat{\psi}_{\mathbf{x}(1)}\nu - \sum_{l=1}^{N}\gamma_l B_{l,r}(1)\right]B_{m,r}(1) + \left[\hat{\psi}_{\mathbf{x}(0)}\nu + \sum_{l=1}^{N}\gamma_l B_{l,r}(0)\right]B_{m,r}(0) \tag{77}$$

Also,

$$0 = \frac{\partial J'}{\partial \beta_m} = \sum_{i=1}^{N}\left\{\sum_{j=0}^{N_q} w_{ij}\hat{F}_{\mathbf{u}}\left[\sum_{l=1}^{N}\gamma_l B_{l,r}(b_{ij})\right]B_{m,r}(b_{ij})\right\} \tag{78}$$

$$0 = \frac{\partial J'}{\partial \gamma_m} = \sum_{i=1}^{N}\left\{\sum_{j=0}^{N_q} w_{ij}\left[\hat{F}(\alpha_k, \beta_k, B_{k,r}(b_{ij})) - \sum_{k=1}^{N}\alpha_k\dot{B}_{k,r}(b_{ij})\right]B_{m,r}(b_{ij})\right\} \tag{79}$$

$$0 = \frac{\partial J'}{\partial \nu} = \hat{\psi}(\alpha_k, B_{k,r}(0), B_{k,r}(1)) \tag{80}$$

Thus, Eqs. (77–80) constitute the KKT conditions $\mathcal{A}_{N\lambda}$. Based on Sec. V.B and using Eq. (67), the discretized first-order optimality conditions $\mathcal{A}_{\lambda N}$ are

$$\begin{gathered}
\sum_{i=1}^{N}\left\{\sum_{j=0}^{N_q} w_{ij}\left[\hat{F}(\alpha_k, \beta_k, B_{k,r}(b_{ij})) - \sum_{k=1}^{N}\alpha_k\dot{B}_{k,r}(b_{ij})\right]B_{m,r}(b_{ij})\right\} = 0 \\
\sum_{i=1}^{N}\left\{\sum_{j=0}^{N_q} w_{ij}\left[\hat{F}_{\mathbf{x}}\sum_{l=1}^{N}\tilde{\gamma}_l B_{l,r}(b_{ij}) + \sum_{l=1}^{N}\tilde{\gamma}_l\dot{B}_{l,r}(b_{ij})\right]B_{m,r}(b_{ij})\right\} = 0 \\
\hat{\Psi}_{\mathbf{x}(1)} + \hat{\psi}_{\mathbf{x}(1)}\nu - \sum_{l=1}^{N}\tilde{\gamma}_l B_{l,r}(1) = 0, \qquad \hat{\psi}_{\mathbf{x}(0)}\nu + \sum_{l=1}^{N}\tilde{\gamma}_l B_{l,r}(0) = 0 \\
\hat{\psi}(\alpha_k, B_{k,r}(0), B_{k,r}(1)) = 0
\end{gathered} \tag{81}$$

for $m = 1, \ldots, N$. Comparing $\mathcal{A}_{N\lambda}$ and $\mathcal{A}_{\lambda N}$, we see that the results from Secs. V.C and V.D hold. The equivalence conditions are

$$\sum_{k=1}^{N}\gamma_k B_{k,r}(0) = -\hat{\psi}_{\mathbf{x}(0)}^T\nu \tag{82}$$

$$\sum_{k=1}^{N}\gamma_k B_{k,r}(1) = [\hat{\Psi}_{\mathbf{x}(1)} + \hat{\psi}_{\mathbf{x}(1)}^T\nu] \tag{83}$$

The costates can be estimated as

$$\hat{\lambda}(t) = \sum_{k=1}^{N}\gamma_k B_{k,r}(t) \tag{84}$$

VII. Remarks

The results presented in this paper not only deal with the optimality analysis of a direct method based on Hilbert space projection, but also provide a framework to analyze the optimality of a more general class of approximation methods, known as methods of weighted residuals (MWRs). This would be of interest because a variety of approximation schemes can be derived from the MWRs for different choices of weight functions [27]. For example, collocation is one such scheme in which the weight functions are chosen as Dirac-δ functions. In this section, we show that some already-established results on costate estimation for the collocation-based methods can be derived using the present approach. Consider the case in which the computational domain $t \in [-1, 1]$ and basis functions $\{\phi_j(t)\}_{j=0}^N$ are the Lagrange interpolating polynomials of degree N. Let $t_k \in [-1, 1]$ be the LGL nodes, so that $t_0 = -1$, $t_N = 1$, and $\{t_l\}_{l=1}^{N-1}$ are the zeros of $\dot{L}_N(t)$. Here, L_N is the Nth degree

Legendre polynomial. It can the be verified that $\{\phi_j(t)\}_{j=0}^N$ satisfy the Kronecker-δ property

$$\phi_j(t_k) = \delta_{ik} \tag{85}$$

Let the equality constraints in Eqs. (16) and (20) be imposed with a slight modification and use the Gauss–Lobbato (GL) integration rule over the grid $\{t_k\}$, so that

$$\frac{1}{w_j}\int_{-1}^{1} \mathcal{R}_F(t)\phi_j(t) \rightarrow \frac{1}{w_j}\sum_{k=0}^{N} w_k\mathcal{R}_F(t_k)\phi_j(t_k) = 0; \quad j = 0, 1, \ldots, N \tag{86}$$

$$\frac{(\tau_f - \tau_0)}{w_j}\int_{-1}^{1} \mathcal{R}_{\mathbf{h}}(t)\phi_j(t) \rightarrow \frac{(\tau_f - \tau_0)}{w_j}\sum_{k=0}^{N} w_k\mathcal{R}_{\mathbf{h}}(t_k)\phi_j(t_k) = 0; \quad j = 0, 1, \ldots, N \tag{87}$$

where w_k are the LGL weights given by

$$w_k = \frac{2}{N(N+1)[L_N(t_k)]^2} \tag{88}$$

Note that Eqs. (16) and (20) are multiplied by $\frac{1}{w_j}$ to obtain the final results in a desired weighted form. Using Eqs. (85–87), we can write

$$\mathcal{R}_F(t_j) = 0; \qquad j = 0, 1, \ldots, N \tag{89}$$

$$\mathcal{R}_{\mathbf{h}}(t_j) = 0; \qquad j = 0, 1, \ldots, N \tag{90}$$

which implies that the residual errors $\mathcal{R}_F(t)$ and $\mathcal{R}_{\mathbf{h}}(t)$ are set to zero at the nodes $\{t_k\}$. This amounts to the satisfaction of the state dynamics and the inequality constraints at the discrete set of points $\{t_k\}_{k=0}^N$. Thus, in this special case, the MHSP reduces to the Legendre pseudospectral method, as described in [11]. Next, we derive the equivalence conditions and costate estimates for this formulation. Taking into account the multiplier $\frac{1}{w_j}$ introduced in Eqs. (86) and (87) and using Eq. (85), the equivalence conditions from Eqs. (52) and (53) take the following form:

$$\sum_{j=0}^{N} \frac{\gamma_j}{w_j}\phi_j(0) = \frac{\gamma_0}{w_0} = -[\hat{\Psi}_{\mathbf{x}(0)} + \hat{\psi}_{\mathbf{x}(0)}^T \nu] \tag{91}$$

$$\sum_{j=0}^{N} \frac{\gamma_j}{w_j}\phi_j(1) = \frac{\gamma_N}{w_N} = [\hat{\Psi}_{\mathbf{x}(1)} + \hat{\psi}_{\mathbf{x}(1)}^T \nu] \tag{92}$$

Using the LGL integration rule, Eq. (54) takes the form

$$\hat{\mathcal{H}}|_{t=t_0} = \hat{\mathcal{H}}|_{t=t_N} = \sum_{j=0}^{N} w_j\hat{\mathcal{H}}|_{t=t_j} \tag{93}$$

Using Eq. (85), the costate estimates are obtained from Eqs. (56–58) as

$$\hat{\lambda}(t_k) = \frac{\gamma_k}{w_k}, \qquad \hat{\mu}(t_k) = \frac{\eta_k}{w_k}, \qquad \upsilon = \nu \tag{94}$$

We see that the conditions in Eqs. (91–93) are identical to the closure conditions derived in [11]. Also, the costate estimates obtained from equation set (94) are same as derived for the Legendre pseudospectral method in [11]. Furthermore, these results are obtained without using any properties of the differentiation matrix employed in [11]. Hence, the present method of costate estimation is more general in nature.

VIII. Numerical Examples

In this section, we use the MHSP to solve three optimal control problems and the results are compared with the known analytical solutions. We present these examples to numerically demonstrate the costate estimation results stated in Theorem 1. The first example has a linear plant with a quadratic cost function. The second example has a nonlinear plant and a terminal cost function. These two problems are solved using both B-splines and Legendre polynomials as basis functions. We would like to clarify that the two approximation schemes are chosen to demonstrate that the MHSP can be used with both local (B-splines) and global (Legendre polynomials) basis functions. The focus is on showing that the costate estimation results of Theorem 1 hold in both the cases. We do not intend to compare the accuracy of one approximation with the other. The third problem demonstrates the efficacy of the MHSP in the presence of path inequality constraints. This problem has a linear plant with a state inequality constraint and is solved by using B-splines as basis functions. All of these examples are programmed in MATLAB® with SNOPT as the NLP solver. The transcription of OCP to NLP is done using OPTRAGEN [30], a MATLAB® toolbox that performs the transcription automatically. The results are compared with the available analytic solutions. The measure for accuracy is defined as the $\%\mathcal{L}_2$ norm of the error with respect to the analytic solution.

A. Example 1: Linear Plant with Quadratic Cost

Minimize

$$J = \frac{1}{2}\int_0^1 (x^2(t) + u^2(t))\,\mathrm{d}t \tag{95}$$

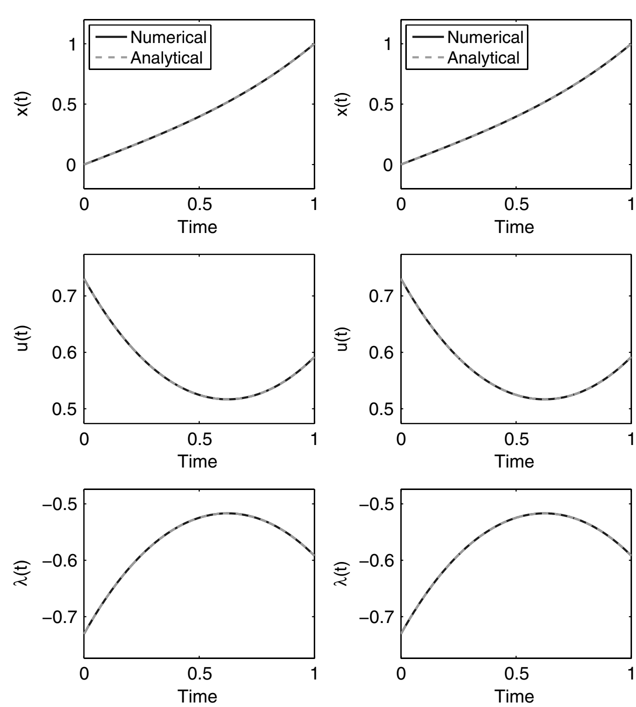
subject to

$$\dot{x}(t) = x(t) + u(t) \qquad x(0) = 0 \qquad x(1) = 1 \tag{96}$$

where $\{x(t), u(t)\}$ is the state-control pair. The analytical solution to this problem is given by [31]

$$x^*(t) = -0.2584e^{-\sqrt{2}t} + 0.2584e^{\sqrt{2}t} \tag{97}$$

$$\lambda^*(t) = -0.6238e^{-\sqrt{2}t} - 0.1070e^{\sqrt{2}t} \tag{98}$$



a) Legendre solution b) B-Spline solution

Fig. 4 Numerical results for example 1 show excellent agreement with the analytical solution. Legendre polynomials are used as basis functions. The costate estimated from the KKT multipliers has a $\%\mathcal{L}_2$ error of the order of 10^{-12}. B-splines are as used as basis functions. The $\%\mathcal{L}_2$ error in the costate estimate is of the order of 10^{-8}. The equivalence conditions hold for both the cases. Solid is numerical, dashed is analytical.

Table 1 Example 1 results

Approximating functions	Legendre polynomials	B-splines
% $\mathcal{L}_2$ error in state $x(t)$	1.2×10^{-14}	3.1×10^{-9}
% $\mathcal{L}_2$ error in control $u(t)$	2.8×10^{-12}	6.8×10^{-8}
% $\mathcal{L}_2$ error in costate $\lambda(t)$	2.9×10^{-12}	2.8×10^{-8}
Equivalence conditions:		
$\sum_{j=1}^{N} \gamma_j \phi_j(0) + [\hat{\Psi}_{\mathbf{x}(0)} + \hat{\psi}^T_{\mathbf{x}(0)} \nu]$	0.3×10^{-6}	0.5×10^{-4}
$\sum_{j=1}^{N} \gamma_j \phi_j(1) - [\hat{\Psi}_{\mathbf{x}(1)} + \hat{\psi}^T_{\mathbf{x}(1)} \nu]$	0.4×10^{-7}	0.2×10^{-4}
Error in cost	4.6×10^{-5}	4.7×10^{-5}

Table 2 Example 2 results

Approximating functions	Legendre polynomials	B-splines
% $\mathcal{L}_2$ error in state $x(t)$	1.5×10^{-7}	1.6×10^{-11}
% $\mathcal{L}_2$ error in control $u(t)$	1.7×10^{-2}	2.8×10^{-4}
% $\mathcal{L}_2$ error in costate $\lambda(t)$	0.1014	1.2×10^{-5}
Equivalence conditions:		
$\sum_{j=1}^{N} \gamma_j \phi_j(0) + [\hat{\Psi}_{\mathbf{x}(0)} + \hat{\psi}^T_{\mathbf{x}(0)} \nu]$	−0.0972	0.0001
$\sum_{j=1}^{N} \gamma_j \phi_j(1) - [\hat{\Psi}_{\mathbf{x}(1)} + \hat{\psi}^T_{\mathbf{x}(1)} \nu]$	−0.0573	0.0040
Error in cost	3.7×10^{-5}	1.5×10^{-7}

$$u^*(t) = -\lambda^*(t) \tag{99}$$

where $\lambda^*(t)$ is the costate associated with the optimal solution. The analytic optimal cost is $J = 0.2959$. This problem is first solved by choosing the basis functions $\{\phi_j(t)\}_{j=1}^N$ as Legendre polynomials with $N = 8$. Next, the solution is obtained using B-splines as approximating functions for which we use fourth-order piecewise polynomials over five intervals with third-degree smoothness at node points. We use the KKT multipliers of the NLP solution to check the equivalence conditions. The costates are estimated using Eq. (57). The results in Fig. 4 show excellent agreement between the analytic solution and the solution obtained from the MHSP. It is evident that the costate estimate obtained directly from the KKT multipliers by using Eq. (57) matches well with the actual costate given by Eq. (98). Table 1 summarizes the results for example 1.

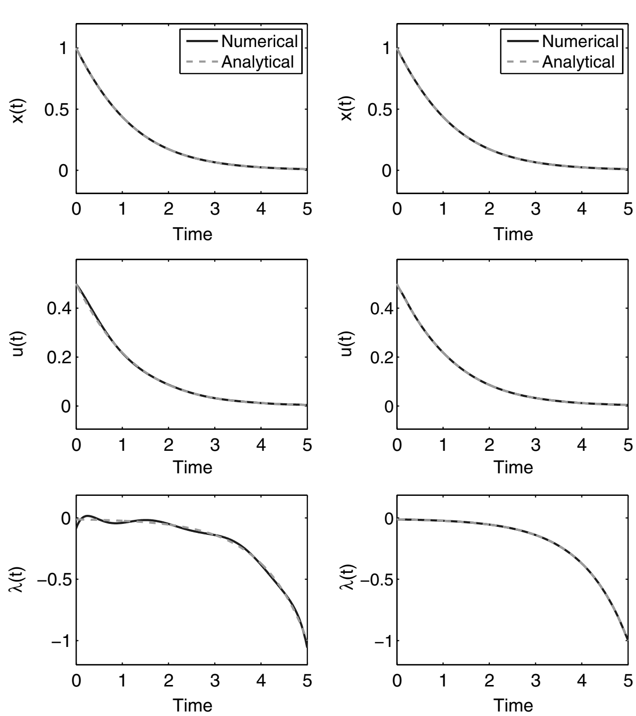
B. Example 2: Nonlinear Plant with Terminal Cost

Minimize

$$J = -x(t_f) \tag{100}$$

subject to

$$\dot{x} = x(t)u(t) - x(t) - u^2(t), \qquad x(t_0) = 1, \qquad t_f = 5 \tag{101}$$



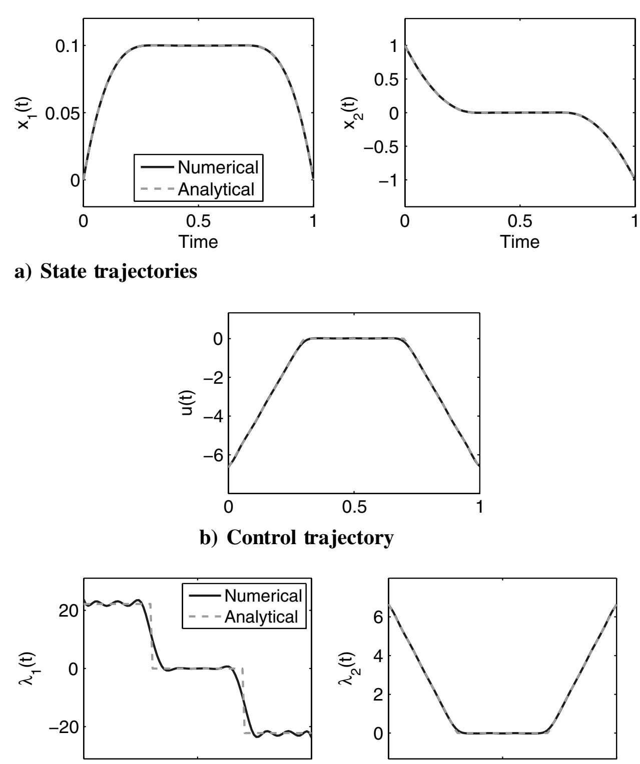
a) Legendre solution b) B-Spline solution

Fig. 5 Comparison of numerical results with the analytical solution for example 2. Legendre polynomials are used as basis functions. The estimated costate has a 0.1014% $\mathcal{L}_2$ error. The equivalence conditions have a large error. B-splines are as used as basis functions. The %$\mathcal{L}_2$ error in the costate estimate is of the order of 10^{-5}. Solid is numerical, dashed is analytical.

where $\{x(t), u(t)\}$ is the state-control pair and $t_f = 5$ is the final time. The analytic solution given by Huntington [32] is

$$x^*(t) = \frac{4}{1+3e^t} \qquad \lambda^*(t) = \frac{-e^{(2ln(1+3e^t)-t)}}{(e^{-5}+6+9e^5)}$$
$$u^*(t) = 0.5x^*(t) \tag{102}$$

where $\lambda^*(t)$ is the costate associated with the optimal solution. The analytic optimal cost is $J = -0.009$. The problem is solved using Legendre polynomials as basis functions with $N = 8$. For the solution obtained by using B-splines as approximating functions, we use twelfth-order piecewise polynomials over seven intervals with sixth-degree smoothness at node points. In both cases, the costates are estimated using Eq. (57). The results are presented in Fig. 5. The $\mathcal{L}_2$ error in the costate estimate is 1.2×10^{-5}% for the B-spline solution and is higher (0.1014%) for the Legendre polynomials solution. The reason Legendre polynomials do not perform well in this example is that the system is highly nonlinear. A higher-order polynomial approximation is required to obtain a better solution. Also, our results show that equivalence conditions must hold to



a) State trajectories

b) Control trajectory

c) Costate trajectories

Fig. 6 Comparison of numerical results with the analytical solution for example 3 (Breakwell problem). B-splines are used as approximating functions for state and control variables. The costates estimated from the KKT multipliers capture the jump discontinuity in costate $\lambda_1(t)$. The equivalence conditions have larger error for $\lambda_1(t)$ than $\lambda_2(t)$. $\mathcal{L}_2$ error in $\lambda_1(t) = 2.2253\%$, $\mathcal{L}_2$ error in $\lambda_2(t) = 0.0116\%$. Solid is numerical, dashed is analytical.

Table 3 Example 3 results

Approximating functions	B-splines
% $\mathcal{L}_2$ error in state $x_1(t)$	0.0001
% $\mathcal{L}_2$ error in control $x_2(t)$	0.0009
% $\mathcal{L}_2$ error in control $u(t)$	0.0122
% $\mathcal{L}_2$ error in costate $\lambda_1(t)$	2.2253
% $\mathcal{L}_2$ error in costate $\lambda_2(t)$	0.0116
Equivalence Conditions:	
$\sum_{j=1}^{N}\gamma_j\phi_j(0)+[\hat{\Psi}_{\mathbf{x}(0)}+\hat{\psi}_{\mathbf{x}(0)}^T\nu]$	$\begin{bmatrix}1.4467\\-0.0751\end{bmatrix}$
$\sum_{j=1}^{N}\gamma_j\phi_j(1)-[\hat{\Psi}_{\mathbf{x}(1)}+\hat{\psi}_{\mathbf{x}(1)}^T\nu]$	$\begin{bmatrix}-1.4467\\-0.0751\end{bmatrix}$
Error in cost	4.9×10^{-4}

obtain costate estimates. However, in the Legendre solution we see that the equivalence conditions have a large error, resulting in erroneous costate estimates. Table 2 summarizes the results for example 2.

C. Example 3: Linear Plant with State Inequality Constraint

Minimize

$$J=\frac{1}{2}\int_0^1 u^2\,\mathrm{d}t \tag{103}$$

subject to

$$\dot{x}_1=x_2 \tag{104}$$

$$\dot{x}_2=u \tag{105}$$

$$x_1(0)=0,\quad x_1(1)=0,\quad x_2(0)=1,\quad x_2(1)=-1 \tag{106}$$

$$x_1(t)\le l=0.1 \tag{107}$$

In the literature [2], this problem is known as the Breakwell problem. We use this problem to demonstrate the applicability of the MHSP in the presence of state inequality constraints, which were not considered in our problem formulation for simplicity. The Breakwell problem has a second-order state inequality constraint. The analytic solution to this problem is given in [2]. The costates associated with the state equations (104) and (105) are denoted as $\lambda_1(t)$ and $\lambda_2(t)$, respectively. The optimal cost is $J=\frac{4}{9l}=4.4444$. The optimal switching structure for this problem is *free–constrained–free*, and the costate $\lambda_1(t)$ has jump discontinuities at times $t_1=3l$ and $t_2=1-3l$. We solve this problem by using B-spline approximation with fourth-order piecewise polynomials over 20 intervals with third-order smoothness. The results are presented in Fig. 6. The states and control trajectories show an excellent match with the analytical solution. The costate estimates capture the jump discontinuity in $\lambda_1(t)$. We see that the equivalence conditions evaluated using Eqs. (52) and (53) have a larger error for $\lambda_1(t)$ compared with $\lambda_2(t)$. Therefore, the $\mathcal{L}_2$ error in the estimate of $\lambda_1(t)$ (2.2253%) is greater than the error in the estimate of $\lambda_2(t)$ (0.0116%). The results are summarized in Table 3.

IX. Conclusions

A direct trajectory optimization technique based on Hilbert space approximation has been presented. In this approach, any set of linearly independent continuous functions can be used to parameterize the state and control trajectories. A set of equivalence conditions are derived under which the difference between a direct and indirect method solution vanishes. These conditions facilitate the estimation of costates from the KKT multipliers associated with the direct optimization. This result provides a generalized framework for the optimality analysis of a wide range of approximation schemes. To demonstrate this claim, the costate estimation results for the Legendre pseudospectral method are shown to be consistent with the results presented by other researchers. We present numerical results on the costate estimation for generalized B-spline parameterization, which show excellent agreement with the analytical solutions. The numerical results demonstrate that the present method can address a fairly general class of optimal control problems, including problems with pure state inequality constraints.

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